# THE $b_r$ - CHROMATIC NUMBER OF A GRAPH AND ITS APPLICATIONS TO SOCIOLOGY

**Noel Marjon E. Yasi, Mary Grace C. Bongabong[1] and Michael P. Baldado Jr.[2]**
[1] Negros Oriental State Univeristy-Siaton Campus, Siaton, Negros Oriental, Philippines
[2] Negros Oriental State Univeristy, Kagawasan Ave., Dumaguete City, Philippines
Email: norsupres@norsu.edu.ph; marygracebongabong@norsu.edu.ph, michael.baldadojr@norsu.edu.ph

***ABSTRACT***: *Let G be a graph and $f: V \rightarrow \{1,2,...,k\}$ be a proper k-coloring of G. We say that f is a $b_r$-coloring if $N(X_i) \cap X_j$ contains at least r-elements for $i \neq j$. If k is the largest number such that G admits a $b_r$-coloring, then k is called the $b_r$-chromatic number of G, denoted by $b_r(G)$.*

*In this paper we introduce the concepts $b_r$-chromatic number of graphs, worm graph, and dream catcher graph, Also, we give their $b_r$-chromatic number. In addition, we present some applications of these notions to sociology.*

**Keywords**: $b_r$-chromatic number; worm graph; dream catcher graph; social networks; intergroup relations

## 1. INTRODUCTION

*Definition 1.* A *vertex coloring* of a graph $G = (V,E)$ is a function $f: V \rightarrow C$ from the set of vertices to a finite set $C$ whose elements are called *colors*.

*Example 1.* Consider graph $G$ in Figure 1. Then $f_1: V \rightarrow \{1,2,3\}$ given by $f_1(a) = f_1(b) = 1$, $f_1(c) = f_1(d) = 2$ and $f_1(e) = f_1(f) = f_1(g) = f_1(h) = 3$ is a vertex coloring of $G$. Similarly, the function $f_2: V \rightarrow \{1,2,3\}$ given by $f_2(a) = 1$, $f_2(f) = f_2(d) = 2$, $f_2(c) = f_2(g) = 3$, and $f_2(b) = f_2(e) = f_2(h) = 4$ is also a vertex coloring of $G$.

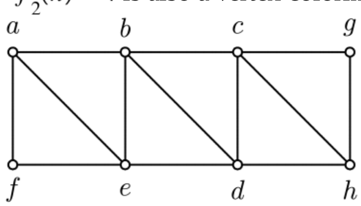


**Figure 1: A graph G**

*Definition 2.* Let $G = (V,E)$ be a simple graph. A surjective vertex labeling $f: V \rightarrow \{1,2,...,k\}$ is called a *k-coloring*.

*Definition 3.* A vertex coloring $f$ of a graph $G = (V,E)$ is *proper* if no two adjacent vertices are assigned the same color.

*Definition 4.* A graph is *k-colorable* if it admits a proper vertex $k$-coloring.

*Definition 5.* The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the smallest $k$ such that $G$ is $k$-colorable.

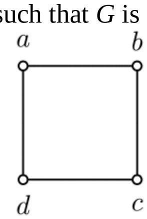


**Figure 2. Graph $C_4$**

*Example 2.* Consider graph $G$ in Figure 1. Then $f_2: V \rightarrow \{1,2,3\}$ given by $f_2(a) = f_2(d) = f_2(g) = 1$; $f_2(f) = f_2(b) = f_2(h) = 2$ and $f_2(e) = f_2(c) = 3$ is a proper vertex coloring of $G$. Moreover, the chromatic number of $G$ is $\chi(G) = 3$.

*Notation 1.* A $k$-coloring $f: V \rightarrow \{1,2,...,k\}$ may also be expressed as a vertex partition $f = \{X_1, X_2, ..., X_k\}$, where $X_i = f^{-1}(i)$. The $X_i$'s are called *color classes*.

*Definition* 6. Let $G = (V,E)$ be a graph and $u \in V$. The *neighborhood* of $u$ is the set $N(u) = \{v : uv \in E\}$. If $X$ is a subset of the vertex set $V$, then the *neighborhood* of $X$ is the set $N(X) = \bigcup_{u \in X} N(u)$.

*Definition 7.* Let $G = (V,E)$ be a graph and $f = \{X_1, X_2, ..., X_k\}$ be a proper $k$-coloring of $G$. We say that $f$ is a $b_r$-coloring if $N(X_i) \cap X_j$ contains at least $r$-elements for $i \neq j$. If $k$ is the largest number such that $G$ admits a $b_r$-coloring, then $k$ is called the $b_r$-*chromatic number* of $G$, denoted by $b_r(G)$.

*Example 3.* Consider the graph $C_4$ in Figure 2. Then $f_3 = \{X_1, X_2\}$ with $X_1 = \{a,c\}$ and $X_2 = \{b,d\}$ is a proper coloring of $C_4$. Note that

$$N(X_1) \cap X_2 = \{b,d\} \cap \{b,d\} = \{b,d\}$$
$$N(X_2) \cap X_1 = \{a,c\} \cap \{a,c\} = \{a,c\}.$$

Hence,

$$|N(X_1) \cap X_2| = 2$$
$$|N(X_2) \cap X_1| = 2.$$

Thus, $f_3$ is a $b_2$-coloring. It can be shown that $k = 2$ is the largest number such that $C_4$ admits a $b_2$-coloring. Therefore, $b_2(C_4) = 2$.

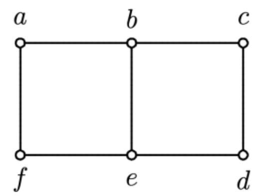


**Figure 3: The graph $P_2 \times P_3$**

*Example 4.* Consider graph $P_2 \times P_3$ in Figure 3. Then $f_4 = \{X_1, X_2\}$ with $X_1 = \{a,c,e\}$ and $X_2 = \{b,d,f\}$ is a proper coloring of $P_2 \times P_3$. Note that

$$N(X_1) \cap X_2 = \{b,d,f\} \cap \{b,d,f\} = \{b,d,f\}$$
$$N(X_2) \cap X_1 = \{a,c,e\} \cap \{a,c,e\} = \{a,c,e\}.$$

Hence,

$$|N(X_1) \cap X_2| = 3$$
$$|N(X_2) \cap X_1| = 3.$$

Thus, $f_4$ is a $b_3$-coloring. It can be shown that $k = 2$ is the largest number such that $P_2 \times P_3$ admits a $b_3$-coloring. Therefore, $b_3(P_2 \times P_3) = 2$.

The *b*-chromatic number [1-20] of a graph represents the maximum number of colors that can be used to properly color its vertices. However, there's an additional twist: each color must have at least one representative vertex adjacent to a vertex of every other color. In other words, we're not just coloring the graph; we're ensuring that each color class has a representative that interacts with all other colors.

The study of the *b*-chromatic number in graph theory was motivated by the desire to explore a more intricate coloring concept beyond the standard chromatic number. Researchers sought to understand how to color graphs while ensuring that each color class has a representative vertex interacting with vertices of all other colors. This additional condition adds complexity and opens up new avenues for investigation, leading to intriguing results and applications in various fields.

In this research paper, the researcher introduced a variety of *b*-chromatic number, called $b_r$-chromatic number. In addition, the researcher introduced a couple of new family of graphs and study their corresponding $b_r$-chromatic numbers. In particular, the researcher provide answers to the following questions. What is the $b_r$-chromatic number of paths, worm graph, cycles, and dream catcher graph,

This study holds significant implications for several critical areas, including the biological networks (specifically protein-protein interaction networks) [21], theoretical computer science [22], and ad hoc and sensor networks [22].

Moreover, the findings of this research promise to deepen our comprehension of *b*-chromatic number and its implications biology and computer science. By bridging current knowledge with future inquiries, this study stands to contribute to a more comprehensive understanding of this vital field.

The concept of the $\boldsymbol{b_1}$**-chromatic number** finds meaningful analogy in sociology, particularly in the study of **social networks, leadership, and intergroup relations**. In a sociogram, individuals form a complex web of connections, and community structures often exhibit overlapping boundaries. A $b_1$-chromatic coloring corresponds to the maximum number of social groups that can coexist such that each has at least one *bridge individual* maintaining ties across all others. This mirrors Granovetter's [23] notion of the *strength of weak ties*, wherein certain actors link disparate social circles, fostering the diffusion of information and innovation.

From the perspective of Bourdieu's [24] theory of *social capital*, the dominant vertex in each color class can be interpreted as an agent endowed with bridging capital—possessing both in-group legitimacy and out-group connectivity. In organizational sociology, this structure resembles Mintzberg's [25] *liaison roles*, which sustain coordination among differentiated units. Thus, a higher $b_1$-chromatic number implies a social configuration that sustains greater functional diversity without fragmenting communication—a mathematical reflection of pluralism with integration.

**Sociological Implications**

- **Social integration:** High $b_1$-chromatic structures model plural societies where diverse groups coexist with strong intergroup links.
- **Leadership and brokerage:** The dominant vertices represent leaders or *brokers* connecting different communities, vital for knowledge transfer and conflict mediation.
- **Cultural diffusion:** In models of opinion dynamics, b-coloring guarantees that every viewpoint has cross-exposure through at least one active intermediary.
- **Institutional design:** The framework helps in visualizing optimal network structures that balance specialization and cohesion within large organizations or communities.

## 2. RESULTS AND DISCUSSION

**2.1 Preliminary Results.** Remark 1 presents a tight upper bound for the $b_r$-chromatic number.

*Remark 1.* Let *G* be a graph of order *n*, and $f = \{X_1, X_2, ..., X_k\}$ be a proper *k*-coloring of *G*. If *f* is a $b_r$-coloring of *G*, then $k \le \frac{n}{r}$.

To see this, let *G* be a graph and $f = \{X_1, X_2, ..., X_k\}$ be a proper *k*-coloring of *G*. If f is a $b_r$-coloring, then $N(X_i) \cap X_j$ contains at least *r*-elements for $i \ne j$. If $N(X_i) \cap X_j$ contains at least *r*-elements for $i \ne j$, then each color class must contain at least *r* vertices. If each color class must contain at least *r* vertices, then *G* must have at least *kr* vertices, that is $kr \le n$. Thus, $k \le \frac{n}{r}$.

Remark 2 presents a tight lower bound for the $b_r$-chromatic number. Since every $b_r$-coloring is a proper coloring, the following remark holds.

Remark 2. Let *G* be a graph of order *n*. Then $\chi(G) \le b_r(G)$.

### 2.2. $\boldsymbol{b_r}$-Chromatic Number of Paths

*Theorem 1.* Let $P_n$ be a path of order *n*. Then $b_{\lfloor \frac{n}{2} \rfloor}(P_n) = 2$

*Proof*. Let $P_n = (v_1, v_2, ..., v_n)$ be a path of order $n$, and consider the labeling $f = \{X_1, X_2\}$ with $X_1 = \{v_1, v_3, ..., v_{n-1}\}$, and $X_2 = \{v_2, v_4, ..., v_n\}$. Note that $N(X_1) = X_2$ and $N(X_2) = X_1$. Hence, $N(X_1) \cap X_2 = X_2 \cap X_2 = X_2$, and $N(X_2) \cap X_1 = X_1 \cap X_1 = X_1$.

Thus, $|N(X_2) \cap X_1| = |X_1 \cap X_1| = |X_1| \geq \lfloor \frac{n}{2} \rfloor$, and $|N(X_1) \cap X_2| = |X_2 \cap X_2| = |X_2| = \lfloor \frac{n}{2} \rfloor$. This implies that $f$ is a $b_{\lfloor \frac{n}{2} \rfloor}$-coloring. Therefore, $b_{\lfloor \frac{n}{2} \rfloor}(P_n) \geq 2$. If $n \geq 4$, then by Remark 1, $b_{\lfloor \frac{n}{2} \rfloor}(P_n) = \dfrac{n}{\lfloor \frac{n}{2} \rfloor} < 2.7$. Accordingly, $b_{\lfloor \frac{n}{2} \rfloor}(P_n) = 2$ .

*Theorem 2*. Let $P_{3k}$ be a path of order $3k$. Then $b_k(P_{3k}) = 3$.

*Proof*. Let $P_{3k} = (v_1, v_2, ..., v_{3k})$ be a path of order $3k$, and consider the labeling $f = \{X_1, X_2, X_3\}$ with $X_r = \{v_i : i \equiv r(mod 3)\}$ for $r = 1,2,3$. Note that $N(X_i) = \bigcup_{i \neq j} X_j$ for $i = 1,2,3$. Hence, $N(X_i) \cap X_j = X_j$ for $i \neq j$. Thus, $|N(X_i) \cap X_j| = |X_j| = k$ for $i = 1,2,3$ and $i \neq j$, implying that $f$ is a $b_k$-coloring. It follows that $b_k(P_{3k}) \geq 3$. But by Remark 1 $b_k(P_{3k}) \leq \dfrac{3k}{k} = 3$. Therefore, $b_k(P_{3k}) = 3$.

### 2.3. $b_r$-Chromatic Number of Worm Graphs

*Definition 5*. Let $P_{rk} = (u_1, u_2, ..., u_{rk})$ be a path of order $rk$. A worm graph $WG^r_{rk} = \{u_1, u_2, ..., u_{rk}\}$ is the graph of order $rk$ obtained from the path $P_{rk}$ by adding edges $u_{s+i}u_{s+j}$ with $i, j \in \{1,2,...,r\}$ for $s = 0,1,2,...,k-1$ excluding $u_{k+1}u_{k+r}$.

*Example 7*. The graph in Figure 3.1 is the worm graph WG4

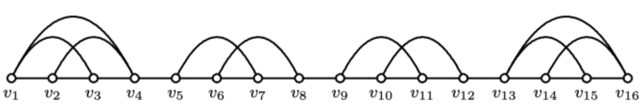


**Figure 4. The worm graph** $WG^4_{16}$

*Theorem 3*. Let $W^r_{kr}$ be a worm graph of order $kr$. Then $b_k(Wr_{kr}) = r$.

*Proof*. Let $WG^r_{rk} = \{v_1, v_2, ..., v_{rk}\}$ be a worm graph of order $kr$, and consider the labeling $f = \{X_1, X_2, ..., X_r\}$ with $X_j = \{v_i : i \equiv j(\text{mod } r)\}$ for $j = 0,1,2,...,r-1$. Note that $N(X_j) = V(W^r_{kr}) \backslash X_j = \bigcup_{i \neq j} X_i$ for $j = 0,1,2,...,r-1$. Hence, $N(X_i) \cap X_j = \left(\bigcup_{i \neq j} X_i\right) \cap X_i = X_i$ for $i \neq j$. Thus, $|N(X_i) \cap X_j| = |X_i| = k$ for $i \neq j$. This implies that $f$ is a $b_k$-coloring. It follows that $b_k(W^r_{kr}) \geq r$. But by Remark 1 $b_k(W^r_{kr}) \leq \dfrac{kr}{k} = r$. Therefore, $b_k(W^r_{kr}) = r$.

### 2.4. $b_r$-Chromatic Number of Cycles

*Theorem 4*. Let $C_{2k}$ be a cycle of order $2k$. Then $b_k(C_{2k})$.

*Proof*. Let $C_{rk} = \{v_1, v_2, ..., v_{rk}\}$ be a cycle of order $rk$, and consider the labeling $f = \{X_1, X_2, ..., X_r\}$ with $X_j = \{v_i : i \equiv j(\text{mod } r)\}$ for $j = 0,1,2,...,r-1$. Note that $N(X_j) = V(C_{rk}) \backslash X_j = \bigcup_{i \neq j} X_i$ for $j = 0,1,2,...,r-1$. Hence, $N(X_i) \cap X_j = \left(\bigcup_{i \neq j} X_i\right) \cap X_i = X_i$ for $i \neq j$. Thus, $|N(X_i) \cap X_j| = |X_i| = k$ for $i \neq j$. This implies that $f$ is a $b_k$-coloring. It follows that $b_k(C_{rk}) \geq r$. But by Remark 1 $b_k(C_{rk}) \leq \dfrac{rk}{k} = r$. Therefore, $b_k(C_{rk}) = r$.

### 2.5. $b_r$-Chromatic Number of Dream Catcher Graphs

*Definition 6*. Let $C_{rk} = \{v_1, v_2, ..., v_{rk}\}$ be a cycle of order $rk$. A dream catcher graph $DC^r_{rk}$ is the graph of order $rk$ obtained from $C_{rk}$ by adding edges by adding edges $u_{s+i}u_{s+j}$ with $i, j \in \{1,2,...,r\}$ for $s = 0,1,2,...,k-1$ excluding $u_{k+1}u_{k+r}$.

*Example 8*. The graph in Figure 5 is the dream catcher graph $DC^5_{15}$.

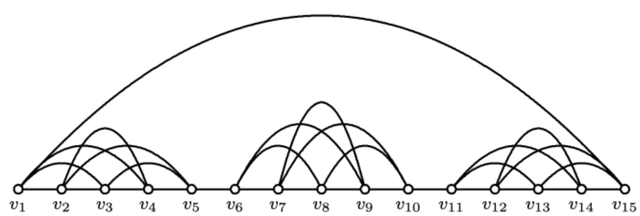


**Figure 5. The dream catcher graph** $DC^5_{15}$

*Theorem 5*. Let $DC^r_{rk}$ be a dream catcher graph of order $rk$. Then $b_k(DC^r_{rk}) = r$.

*Proof*. Let $DC^r_{rk} = \{v_1, v_2, ..., v_{rk}\}$ be a dream catcher graph of order $rk$, and consider the labeling $f = \{X_1, X_2, ..., X_r\}$ with $X_j = \{v_i : i \equiv j(\text{mod } r)\}$ for $j = 0,1,2,...,r-1$. Note that $N(X_j) = V(DC^r_{rk}) \backslash X_j = \bigcup_{i \neq j} X_i$ for $j = 0,1,2,...,r-1$. Hence, $N(X_i) \cap X_j = \left(\bigcup_{i \neq j} X_i\right) \cap X_i = X_i$ for $i \neq j$. Thus, $|N(X_i) \cap X_j| = |X_i| = k$ for $i \neq j$. This implies that $f$ is a $b_k$-coloring. It follows that $b_k(DC^r_{rk}) \geq r$. But by Remark 1 $b_k(DC^r_{rk}) \leq \dfrac{rk}{k} = r$. Therefore, $b_k(DC^r_{rk}) = r$.

**3. ACKNOWLEDGMENT**

The authors would like to thank *Negros Oriental State University* through the *Rural Engineering and Technology Center* for the technical support.

**REFERENCES:**

[1] M. Alkhateeb and A. Kohl. Upper bounds on the *b*-chromatic number and results for restricted graph classes. Discussiones Mathematicae Graph Theory, 31(4):709–735, 2011.

[2] N. Ansari, R. Chandel, and R. Jamal. On *b*-chromatic number of prismgraph families. Applications and Applied Mathematics: An I nternational Journal (AAM), 13(1):20, 2018.

[3] R. Balakrishnan and S. F. Raj. Bounds for the *b*-chromatic number of $g-v$. Discrete Applied Mathematics, 161(9):1173–1179, 2013.

[4] R. Balakrishnan, S. F. Raj, and T. Kavaskar. *b*-chromatic number of cartesian product of some families of graphs. Graphs and Combinatorics, 30(3):511–520, 2014.

[5] M. Blidia, F. Maffray, and Z. Zemir. On *b*-colorings in regular graphs. Discrete Applied Mathematics, 157(8):1787–1793, 2009.

[6] S. Cabello and M. Jakovac. On the *b*-chromatic number of regular graphs. Discrete Applied Mathematics, 159(13):1303–1310, 2011.

[7] R. Chandel, R. Jamal, and N. Ansari. The *b*-chromatic number of ladder graph. J˜n–an–abha, page 139, 2017.

[8] B. Effantin. The *b*-chromatic number of power graphs of complete caterpillars. Journal of Discrete Mathematical Sciences and Cryptography, 8(3):483–502, 2005.

[9] B. Effantin and H. Kheddouci. The *b*-chromatic number of power graphs. Discrete Mathematics & Theoretical Computer Science, 6, 2003.

[10] Dearlove, J. (1997). The academic labour process: From collegiality and professionalism to managerialism and proletarianisation. *Higher Education Review*, *30*(1), 56-75.

[11] A. El Sahili, H. Kheddouci, M. Kouider, and M. Mortada. The *b*-chromatic number and *f*-chromatic vertex number of regular graphs. Discrete Applied Mathematics, 179:79–85, 2014.

[12] J. L. Gross and J. Yellen. Handbook of graph theory. CRC press, 2003.

[13] R. W. Irving and D. F. Manlove. The *b*-chromatic number of a graph. Discrete Applied Mathematics, 91(1-3):127–141, 1999.

[14] M. Jakovac and I. Peterin. On the *b*-chromatic number of some graph products. Studia Scientiarum Mathematicarum Hungarica, 49(2):156–169, 2012.

[15] M. Kouider and A. El Sahili. About *b*-colouring of regular graphs. Rapport de Recherche, 1432, 2006.

[16] M. Kouider and M. Mah´eo. Some bounds for the *b*-chromatic number of a graph. Discrete Mathematics, 256(1-2):267–277, 2002.

[17] J. Kratochv´ıl, Z. Tuza, and M. Voigt. On the b-chromatic number of graphs. In Graph-Theoretic Concepts in Computer Science: 28th International Workshop, WG 2002 Cesk`y Krumlov, Czech Republic, June 13–15, 2002 Revised Papers 28, pages 310–320. Springer, 2002.

[18] P. Lisna and M. Sunitha. b-chromatic sum of a graph. Discrete Mathematics, Algorithms and Applications, 7(04):1550040, 2015.

[19] P. Lisna and M. Sunitha. A note on the *b*-chromatic number of corona of graphs. Journal of Interconnection Networks, 15(01n02):1550004, 2015.

[20] Z. Masih and M. Zaker. Some comparative results concerning the grundy and b-chromatic number of graphs. Discrete Applied Mathematics, 306:1–6, 2022.

[21] L. Jaffke and P. T. Lima. A complexity dichotomy for critical values of the b-chromatic number of graphs. Theoretical Computer Science, 815:182–196, 2020.

[22] M. Kouider and M. Mah´eo. The *b*-chromatic number of the cartesian product of two graphs. Studia Scientiarum Mathematicarum Hungarica, 44(1):49–55, 2007.

[23] P. Bourdieu. *The forms of capital*. In J. Richardson (Ed.), *Handbook of theory and research for the sociology of education* (pp. 241–258). Greenwood Press, 1986.

[24] M. S. Granovetter. The strength of weak ties. *American Journal of Sociology, 78*(6),1360-1380, 1973.

[25] H. Mintzberg. *The structuring of organizations: A synthesis of the research*. Prentice-Hall, 1979.